Indefinite Metric and Stationary External Interactions of Quantized Fields*

BERT SCHROER
Department of Physcis, University of Pittsburgh, Pittsburgh, Pennsylvania 15213

AND

JORGE ANDRE SWIECA
Departamento de Fisica, Universidade de Sao Paulo, Sao Paulo, Brazil

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We discuss the necessary modifications of the state space for quantum fields interacting with strong stationary external fields. In addition to the introduction of an indefinite metric, we discuss the possibility of having a positive-definite metric without a vacuum state.

I. INTRODUCTION

IN the preceding paper[1] we discussed the formal and physical aspects of time-dependent external interactions of quantized fields. Whereas in the time-dependent case (i.e., the interaction is "switched on and off"), the underlying Hilbert space is the free-particle Fock space of the particles before (or after) the interaction and hence, the metric is positive definite,[1] this ceases to be the case for time-independent (i.e., stationary) external interactions. Such interactions either require the introduction of an indefinite metric or (with a positive-definite metric) lead to a breakdown of the vacuum postulate and a breakdown at a complete particle interpretation. This happens if the interaction becomes sufficiently strong. The mathematical aspect of this peculiar phenomenon is completely consistent, and physically there is no catastrophic instability[2] and therefore no *a priori* reason to reject such a possibility. The problem of whether realistic quantum field theories can lead to such solutions is not discussed here, and we will only make some speculative remarks. We want to emphasize, however, that the external interaction of a spin-$\frac{1}{2}$ particle is an exception; here the conventional picture of positive-definite-metric bound states remains correct independent of the size of the interaction, a fact already pointed out by Schiff, Snyder, and Weinberg.[3]

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[1] B. Schroer, R. Seiler, and A. J. Swieca, preceding paper, Phys. Rev. D 2, 2927 (1970).

[2] The word instability is used in a generic sense to denote various types of physical and mathematical pathologies and inconsistencies. These may already occur on the c-number (or algebraic) level, like the inconsistencies of Pauli-Fierz equations with particular external interactions (Ref. 1), or the coupling of unphysical solutions to the physical ones, as in the case of Joos-Weinberg equations with interactions (Ref. 1). On the other hand, instabilities may show up on the level of quantization as exemplified by the "old" Dirac theory (without the reinterpretation) of quantized positive- and negative-energy electrons in interaction with photons. The instability relevant for this paper is the one discussed by Schiff, Snyder, and Weinberg (Ref. 5) of a Klein-Gordon particle in a smooth external potential if one quantizes the Klein-Gordon field in a positive-definite Fock space. As pointed out by these authors, a potential sufficiently large numerically, but otherwise well behaved, leads to a mathematical inconsistency for such a quantization.

[3] L. I. Schiff, H. Snyder, and J. Weinberg, Phys. Rev. 57, 315 (1940).

II. SIMPLE EXAMPLE FOR $s=0$ WITH INDEFINITE METRIC

Consider a charged Klein-Gordon field with an external scalar interaction $V(\mathbf{x})$:

$$\left[\frac{\partial^2}{\partial t^2}-\nabla^2+m(m-V)\right]A(x)=0. \tag{1}$$

The corresponding stationary classical equation

$$E^2\phi_E=\{-\nabla^2+m[m-V(\mathbf{x})]\}\phi_E \tag{2}$$

has (for a suitable class of potentials, for example, Kato[4] potentials) a complete set of discrete and continuous eigenstates where the latter correspond to scattering states. Since the energy enters the equation quadratically, we get for each energy also the negative energy. The energy becomes pure imaginary if the potential is sufficiently attractive. We normalize the wave functions according to ($E^2=\mathbf{k}^2+m^2$)

$$\int\bar{\Phi}_i\Phi_i d^3x=1 \quad \text{for bound-state wave functions},$$

$$\int\bar{\Phi}_\mathbf{k}(\mathbf{x})\Phi_{\mathbf{k}'}(\mathbf{x})d^3x=\delta(\mathbf{k}-\mathbf{k}') \quad \text{for scattering state wave functions}. \tag{3}$$

The completeness of these wave functions is conveniently expressed as

$$\sum_i \Phi_i(\mathbf{x})\bar{\Phi}_i(\mathbf{y})+\int d^3k\, \Phi_\mathbf{k}(\mathbf{x})\bar{\Phi}_\mathbf{k}(\mathbf{y})=\delta(\mathbf{x}-\mathbf{y}). \tag{4}$$

We write the field operator at $t=0$ as

$$A(\mathbf{x})=\sum q_i\Phi_i(\mathbf{x})+\int d^3k\, q(\mathbf{k})\Phi_\mathbf{k}(\mathbf{x}),$$
$$\dot{A}^\dagger(\mathbf{x})\equiv\pi(\mathbf{x})=\sum_i p_i\bar{\Phi}_i(\mathbf{x})+\int d^3k\, p(\boldsymbol{k})\bar{\Phi}_\mathbf{k}(\mathbf{x}). \tag{5}$$

[4] For example, T. Kato, *Perturbation Theory for Linear Operators* (Springer-Verlag, New York, 1968).

The canonical commutation relations,

$$[A(\mathbf{x}),\pi(\mathbf{y})]=i\delta(\mathbf{x}-\mathbf{y})=-[A^+(\mathbf{x}),\pi^+(\mathbf{y})],$$
$$[A(\mathbf{x}),A(\mathbf{y})]=0=[\pi(\mathbf{x}),\pi(\mathbf{y})]=\text{etc.},\tag{6}$$

together with the completeness relation (4), lead to

$$[q_i,p_j]=i,$$
$$[q(\mathbf{k}),p(\mathbf{k}')]=i\delta(\mathbf{k}-\mathbf{k}'),\tag{7}$$
$$[q_i,q_j]=0=\text{etc.}$$

For all continuous states[5] and bound states whose energy is real, we introduce the "mode" operators

$$q(k)=\frac{1}{(2\omega_k)^{1/2}}[b^\dagger(\mathbf{k})+a(\mathbf{k})],$$
$$p(\mathbf{k})=-i(\tfrac{1}{2}\omega_k)^{1/2}[b(\mathbf{k})-a^\dagger(\mathbf{k})],$$
$$q_i=\frac{1}{(2\omega_i)^{1/2}}(b_i^\dagger+a_i),\tag{8}$$
$$p_i=-i(\tfrac{1}{2}\omega_i)^{1/2}(b_i-a_i^\dagger),$$

whereas for the imaginary-energy bound-state wave functions we introduce the transformation ($\lambda^2=-E^2$)

$$q_i=\frac{1}{(2\lambda_i)^{1/2}}(a_i+b_i),$$
$$q_i^\dagger=\frac{1}{(2\lambda_i)^{1/2}}(a_i^\dagger+b_i^\dagger),\tag{9}$$
$$p_i=(\tfrac{1}{2}\lambda_i)^{1/2}(b_i^\dagger-a_i^\dagger),$$
$$p_i^\dagger=(\tfrac{1}{2}\lambda_i)^{1/2}(b_i-a_i).$$

The operator at an arbitrary time can then be written as

$$A(\mathbf{x},t)=\sum_i \frac{a_i}{(2\lambda_i)^{1/2}}\Phi_i(\mathbf{x})e^{-\lambda_i t}+\frac{b_i}{(2\lambda_i)^{1/2}}\Phi_i(\mathbf{x})e^{\lambda_i t}$$
$$+\sum_{E_i\geq 0}(a_i e^{-iE_it}+b_i^\dagger e^{iE_it})\frac{\Phi_i(\mathbf{x})}{(2E_i)^{1/2}}$$
$$+\int\left(\frac{a(k)}{(2\omega_k)^{1/2}}\Phi_k(\mathbf{x})e^{-i\omega_k t}+\frac{b^\dagger(k)}{(2\omega_k)^{1/2}}\Phi_k(\mathbf{x})e^{i\omega_k t}\right)d^3k.\tag{10}$$

One might think that the quantization will lead to instabilities owing to the presence of the exponential increasing time factor. This is, however, not the case. It will be demonstrated that there are precisely two possibilities of "quantization," i.e., construction of a Hilbert space with the operators a, b acting in it as an irreducible set of operators which are consistent with the given Hamiltonian. The first quantization consists in viewing the imaginary-energy operators in the same way as the creation and annihilation operators of the real energy modes. However, the commutation relations for the imaginary-energy modes are

$$[a,a]=[a^\dagger,a]=[b^\dagger,b]=[b,b]=0,\tag{11}$$
$$[a,b^\dagger]=i.$$

These follow from the definition (9) and the commutation relation (7). The only consistent way for constructing a Fock space is to use an indefinite metric:

$$a|0\rangle=b|0\rangle=0,$$
$$a^\dagger|0\rangle=|a\rangle,\qquad b^\dagger|0\rangle=|b\rangle,\tag{12}$$
$$\langle a|a\rangle=\langle b|b\rangle=0,\quad \langle a|b\rangle=i.$$

The Hamiltonian which does the time translation (10) is then

$$H=i\sum_i\lambda_i(\hat{N}_{ai}-\hat{N}_{bi})+\sum_{E_i>0}E_i(N_{ai}+N_{bi})$$
$$+\int\omega(k)[N_a(\boldsymbol{k})+N_b(\boldsymbol{k})]d^3k,\tag{13}$$

where the $N(k)$ and N_i for the positive-energy modes are the usual number operators, and for the imaginary-energy modes

$$\hat{N}_{a_i}=ia_i^\dagger b_i,\tag{14}$$
$$\hat{N}_{b_i}=-ib_i^\dagger a_i.$$

The charge operator Q which has the infinitesimal commutation relation $[Q,A(x)]=-A(x)$ is given by

$$Q=\sum_i(-\hat{N}_{ai}-\hat{N}_{bi})+\sum_{i,E_i>0}(-N_{ai}+N_{bi})$$
$$+\int[-N_a(\boldsymbol{k})+N_b(\boldsymbol{k})]d^3k.\tag{15}$$

Up to (infinite) c numbers, these operators are the same as their well-known expression in terms of canonical variables:

$$H=\int\pi^\dagger(\boldsymbol{x})\pi(\mathbf{x})+\nabla A^\dagger(\boldsymbol{x})\nabla A(\mathbf{x})$$
$$+m(m-V)A^\dagger(\boldsymbol{x})A(\mathbf{x})d^3x,\tag{16a}$$

$$Q=-i\int\pi(\boldsymbol{x})A(x)-\pi^\dagger(\boldsymbol{x})A^\dagger(x)d^3x.\tag{16b}$$

The case of bound states with zero energy in our model requires a special treatment. In this case, the contributions of these modes to the Hamiltonian (16a) and to

[5] The continuous spectrum is completely contained in $E^2\geq m^2$. For our special model (2), this follows immediately from the location of the continuous spectrum of the Schrödinger equation. For the general case, it is a special result of the statement that the part of the resolvent (41) outside the continuum $E^2\geq m^2$ is a compact operator, a fact which is demonstrated in a similar manner as in the Schrödinger theory (Ref. 4).

the charge operator (16b) are

$$\hat{H}=p^\dagger p\,, \tag{17a}$$

$$\hat{Q}=-i(pq-p^\dagger q^\dagger)\,. \tag{17b}$$

These operators still commute as they should: $[\hat{H},\hat{Q}]=0$. Introducing as our creation and annihilation variables

$$a^\dagger=p^\dagger\,,\quad b^\dagger=iq\,,$$

we obtain

$$[a,a^\dagger]=0=\cdots\,,\quad [b^\dagger,a]=-1\,, \tag{18a}$$

$$\hat{H}=a^\dagger a\,, \tag{18b}$$

$$\hat{Q}=i(b^\dagger a-a^\dagger b)+c\text{-number}\,, \tag{18c}$$

with

$$\hat{Q}|a\rangle=-|a\rangle\quad (c\text{-number omitted})\,, \tag{19a}$$

$$\hat{Q}|b\rangle=-|b\rangle \tag{19b}$$

and

$$\hat{H}|a\rangle=0\,, \tag{20a}$$

$$\hat{H}|b\rangle=|a\rangle\,; \tag{20b}$$

i.e., $|a\rangle$ is an eigenstate of $\hat{H}$ with eigenvalue zero, whereas $|b\rangle$ is an *associated* eigenvector to the same eigenvalue. In terms of physicists' language about the indefinite metric in quantum theory, this is the so-called dipole-ghost situation.

It should be emphasized that the conservation of energy and charge prevents any catastrophic pair creation of imaginary-energy particles. This would not be the case if we were to view our stationary problem as an adiabatic limit of time-dependent problems. In this case, we would always stay in a positive-definite metric space, and we would run into the "Klein paradox."[6] The vacuum of our genuinely stationary problem is infinitely different (for example, in particle number) from the "adiabatic" vacuum. The expectation values of the current density and the energy density vanish in the imaginary-energy states $|a\rangle$ and $|b\rangle$, respectively.

III. SAME INTERACTION IN THE "NO-VACUUM REPRESENTATION"

As far as the treatment of the positive-energy operators is concerned, the Fock representation is the only possibility. The contribution from the imaginary-energy states can, however, also be represented differently from the (necessarily indefinite) Fock representation of Sec. II. In order to see this, let us introduce instead of a_i, b_i the canonical variables (we omit for convenience the index i)

$$p=(p^{(1)}+ip^{(2)})/\sqrt{2}\,,\quad q=(q^{(1)}-iq^{(2)})/\sqrt{2}\,. \tag{21}$$

In terms of these Hermitian quantities, the contribution of the imaginary-energy operators to the Hamiltonian can be written as

$$p^+p-\lambda^2q^+q=\tfrac{1}{2}[(p^{(1)})^2-\lambda^2(q^{(1)})^2]+\tfrac{1}{2}[(p^{(2)})^2-\lambda^2(q^{(2)})^2]\,. \tag{22}$$

These operators clearly have a representation in a positive-definite Hilbert space; the Hamiltonian (22) describes "repulsive oscillators."

It is convenient for our discussion to introduce a function space $L^2(x,y)$ and to write our operators as

$$\begin{aligned} p^{(1)}&=(\tfrac{1}{2}\lambda)^{1/2}\left(x-i\frac{\partial}{\partial x}\right),\\ q^{(1)}&=\frac{1}{(2\lambda)^{1/2}}\left(x+i\frac{\partial}{\partial x}\right),\\ p^{(2)}&=(\tfrac{1}{2}\lambda)^{1/2}\left(y-i\frac{\partial}{\partial y}\right),\\ q^{(2)}&=\frac{1}{(2\lambda)^{1/2}}\left(y+i\frac{\partial}{\partial y}\right). \end{aligned} \tag{23}$$

Then the contribution to the Hamiltonian and the charge operator, (16a) and (16b), can be written

$$\hat{H}=\tfrac{1}{2}\lambda\left(ix\frac{\partial}{\partial x}+\tfrac{1}{2}i\right)+\tfrac{1}{2}\lambda\left(iy\frac{\partial}{\partial y}+\tfrac{1}{2}i\right), \tag{24a}$$

$$\hat{Q}=-i\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right). \tag{24b}$$

Equation (24a) is proportional to the infinitesimal generator of the dilatation operator:

$$U(a)\psi(x,y)=e^{-a}\psi(e^{-a}x,e^{-a}y)\,,\qquad U(a)=\exp(ia\hat{H}/\lambda)\,, \tag{25}$$

whereas $\hat{Q}$ is the infinitesimal generator of rotations in the xy plane. The simultaneous eigenfunctions are

$$\psi_{\epsilon,m}(r,\phi)=\frac{(r)^{i\epsilon-1}}{(2\pi)^{1/2}}\frac{e^{im\phi}}{(2\pi)^{1/2}}\,,\quad \epsilon=\text{real}\,,\quad m=\text{integer} \tag{26}$$

$$\epsilon=\text{eigenvalue of dilatation generator,}$$

$$m=\text{eigenvalue of }\hat{Q}.$$

Then eigenfunctions are orthonormal,

$$\int\bar{\psi}_{\epsilon'm'}\psi_{\epsilon,m}r\,dr\,d\phi=\delta(\epsilon-\epsilon')\delta_{mm'}\,, \tag{27a}$$

and form a complete set:

$$\int_{-\infty}^{\infty}d\epsilon\,\psi_{\epsilon,m}(\mathbf{x}_1)\bar{\psi}_{\epsilon,m}(\mathbf{x}_2)=\frac{\delta(r_1-r_2)}{r_1}\delta(\phi_1-\phi_2)=\delta^2(\mathbf{x}_1-\mathbf{x}_2)\,. \tag{27b}$$

[6] This is discussed in most of the standard text books on quantum field theory, for example, in J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, (1968).

Thus, we have constructed a field $A(x)$ fulfilling Eq. (1) and operators H, Q acting in the Hilbert space:

$$\mathcal{H}=\mathcal{H}_{\text{Fock}}\otimes\prod_i L^2(x_i,y_i)\,, \tag{28}$$

where $\mathcal{H}_{\text{Fock}}$ is the Fock space generated by the positive-energy creation and annihilation operators. Clearly, H and Q have the correct commutation relations with $A(\mathbf{x})$, and hence

$$A(\mathbf{x},t)=e^{iHt}A(\mathbf{x})e^{-iHt}.$$

The field $A(x)$ is a causal solution of the external field equation, but in $\mathcal{H}$ there is no vacuum state. The part due to the harmonic-repulsive-oscillator treatment is essentially a "no particle"-like solution. We will not discuss the conceptual difficulties with an asymptotic-particle interpretation which occur in such a case.

The fact that the two types of quantization we discovered are the only possible ones for the Hamiltonian problem at hand follows from the well-known fact that the representation of the continuous part of the Hamiltonian which has the form

$$\int\omega(\boldsymbol{k})[N_a(\boldsymbol{k})+N_b(\boldsymbol{k})]d^3k$$

uniquely selects one representation, namely, the Fock representation of the $a(\boldsymbol{k})$ and $b(\boldsymbol{k})$ if the Hamiltonian is to make sense as a self-adjoint operator in the representation space.[7] The remaining dynamical variables related to the bound-state (real or complex) energy c-number wave functions contribute a finite sum of "inverted" oscillators (19) to the Hamiltonian. For the usual quantum-mechanical state space, this leads to the no-vacuum quantization. The other (indefinite) quantization can only be obtained by abandoning the usual structure of quantum theory in favor of an indefinite metric. The states of this indefinite-metric quantization can be formally obtained by analytically continuing the ordinary oscillator eigenfunctions in the spring constant to imaginary values. The uniqueness of the representation of the dynamical problem in the positive-definite-metric case is an illustration of a conjecture by Araki[8] that for dynamical problems not involving zero mass, the algebraic form of the Hamiltonian determines uniquely an irreducible representation of the canonical commutative relations.

IV. REMARKS CONCERNING GENERAL CASE

The discussion can be generalized to other interactions and also to higher-spin equations. Consider, for example, an $s=0$ particle interacting with an external electromagnetic field:

$$(D_\mu D^\mu+m^2)A(x)=0\,,\quad D_\mu=\partial_\mu-iA_\mu. \tag{29}$$

[7] S. Doplicher Commun. Math. Phys. **3**, 228 (1966).
[8] H. Araki, J. Math. Phys. **1**, 492 (1960).

The corresponding c-number equation in the two-component formalism reads

$$i\partial_t\psi=(H_0+H_i)\psi\,,\qquad \psi=\begin{pmatrix}\Phi\\ \dot\Phi\end{pmatrix},$$

$$H_0=i\begin{pmatrix}0 & 1\\ -K^2 & 0\end{pmatrix},\quad K=(1-\Delta)^{1/2}, \tag{30}$$

$$H_1=\begin{pmatrix}V & 0\\ \mathbf{A}^2+i\{\partial_l,A^l\} & V\end{pmatrix}.$$

If we were able to find a complete set of eigenstates

$$H\psi_E=E\psi_E\,, \tag{31}$$

we would write the field as

$$\Psi^{\text{op}}\equiv\begin{pmatrix}A(\mathbf{x})\\ \pi^\dagger(\mathbf{x})\end{pmatrix}=\sum_i\psi_{Ei}(\boldsymbol{x})C_i$$

$$+\int_{E>m;\,E<-m}\psi_{E,k}(\boldsymbol{x})C(\boldsymbol{k})d^3k\,, \tag{32}$$

$$\pi^\dagger=D_0A\,,$$

where we have used C as a generic notation for "annihilation" ("creation") operators for particles (antiparticles).

The canonical structure of the field is now expressed as

$$[\Psi^{\text{op}}(\mathbf{x}),\Psi^{+\text{op}}(\mathbf{y})]=-\tau_2\delta(\mathbf{x}-\mathbf{y})\,. \tag{33}$$

The physical norm (from conserved current) of the two-component wave function is the time-independent but indefinite expression

$$(\psi,\psi)=-\int\psi^\dagger(x)\tau_2\psi(x)d^3x. \tag{34}$$

The commutation relation of the operators C follow from the normalization properties of the energy eigenfunctions ψ_E in the physical metric (34), together with the canonical structure (33). Before we work out those, let us make some salient remarks about completeness of eigenstate (31). The c-number Hamiltonian H is formally self-adjoint with respect to the indefinite metric (34); this is nothing but the conservation law of the c-number "charge." But this "pseudo"- self-adjointness does not lead to the completeness of eigenstates. In fact, from the analogy with general matrices and Hilbert-Schmidt kernels, one might hope that the eigenstates, together with the associated eigenstates, form a complete set. A state $\psi_E{}^a$ is called an associated eigenvector of H if

$$H\psi_E{}^a=E\psi_E{}^a+\psi_E\,, \tag{35}$$

where ψ_E is an ordinary eigenstate of H. Note that the

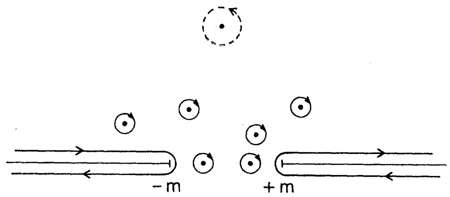

FIG. 1. Singularities of the resolvent in the complex energy plane, and the contour of integration for the proof of completeness.

eigenvalues of pseudo-self-adjoint problems like (31) do not have to be real. However, the pseudo-adjointness with respect to the nondegenerate metric (34) brings about the following properties:

(1) The eigenstates belonging to an eigenvalue E_1 are orthogonal [in the sense of the inner product (34)] to eigenstates belonging to E_2 if $\bar{E}_2 \neq E_1$.

(2) For each complex eigenvalue E there exists the complex conjugate value $\bar{E}$. The eigenstate belonging to a complex eigenvalue necessarily has vanishing physical norm, and the inner product between $\psi_{\bar{E}}$ and ψ_E can be chosen as

$$(\psi_{\bar{E}},\psi_E)=i. \tag{36}$$

(3) If an associated eigenvector $\psi_E{}^a$ exists, then the norm of the eigenvector must vanish. Changing the associated eigenvector by adding a multiple of the eigenvector, one can always obtain (for the changed associated vector)

$$(\psi_E{}^a,\psi_E{}^a)=0\,,\quad (\psi_E{}^a,\psi_E)=1\,. \tag{37}$$

The three statements are demonstrated in a straightforward way by using the pseudo-self-adjointness of H with respect to the nondegenerate metric (31).

The proof of the completeness of eigenstates (including associated eigenstates) is considerably more involved. Here the indefinite metric is of no use and one has to employ the positive-definite energy metric:

$$\|\psi\|_E{}^2=(K\Phi,K\Phi)+(\dot{\Phi},\dot{\Phi})\,, \tag{38}$$

where

$$(\chi,\chi)\equiv\int|\chi|^2d^3x\,.$$

In this metric H_0 is a self-adjoint operator, and H_1 is bounded[1] (but not self-adjoint). The resolvent $1/(H_0-z)$ exists for all z away from the cuts $(-\infty,-m)$, $(m,+\infty)$, and the kernel $G_z(\mathbf{x}-\mathbf{x}')$ representing x space is a function which falls off for large distances. For interactions which are bounded decreasing functions in x space, the operator $[1/(H_0-z)]H_1$ is Hilbert-Schmidt (H.S.) in the energy norm. Its H.S. norm is the "energy" trace of

$$H_1{}^\dagger\frac{1}{H_0-z}\,\frac{1}{H_0-z}H_1$$

and can be made arbitrarily small for Imz sufficiently large. Because of this the resolvent

$$\frac{1}{H-z}=\frac{1}{H_0-z}\left(1+H_1\frac{1}{H_0-z}\right)^{-1} \tag{39}$$

exists for sufficiently large Imz and is analytic. The completeness is shown by writing a contour integral around an analytic point

$$\oint\frac{1}{H-z}dz=0\,. \tag{40}$$

One then deforms the contour in the way indicated in Fig. 1. In order to make such a shift of contour possible, we must write the resolvent in terms of a H.S. operator which retains the H.S. property on the boundary:

$$\begin{aligned}\frac{1}{H-z}&=\frac{1}{H_0-z}\left(1-H_1\frac{1}{H_0-z}\right.\\&\qquad\left.+H_1\frac{1}{H_0-z}AB\frac{1}{H_0-z}-\cdots\right) \qquad (41)\\&=\frac{1}{H_0-z}\left(1-H_1\frac{1}{H_0-z}+A\frac{C(z)}{1+C(z)}A\frac{1}{H_0-z}\right),\end{aligned}$$

with

$$C(z)=B\frac{1}{H_0-z}B\,.$$

Here we imagine the interaction being written as

$$H_1=A\cdot B=BA\,,$$

where A and B are matrices whose matrix elements contain the square root of the local interaction. In the special case of an electric potential, A and B are just the ordinary square roots

$$A=B=\begin{pmatrix}V^{1/2} & 0\\ 0 & V^{1/2}\end{pmatrix}. \tag{42}$$

The two B's provide enough convergence so that $C(z)$ will stay H.S. even on the cut.[9] The only difference from the self-adjoint case lies in the fact that $C(z)$ is a *non-self-adjoint H.S. operator.* The eigenstates and associated eigenstates of $C(z)$ with the eigenvalue -1 give a singularity of $[1+C(z)]^{-1}$. We now can shift the contour (37) to the cuts. The pole contribution will be exactly the sum over the eigenspaces (including the associated vectors). The contribution from infinity corresponds to the δ function in the completeness relation, whereas the cut is just the contribution from the (scattering) continuum. Note that the number of associated

[9] See, for example, W. Hunziker, in *Lectures in Theoretical Physics,* edited by W. E. Brittin, B. W. Downs, and J. Downs (Wiley-Interscience, New York, 1967), Vol. IX.

and ordinary eigenvectors (the so-called algebraic multiplicity) belonging to the eigenvalue 1 is finite. We will be satisfied here with this brief sketch and give a more explicit proof elsewhere. Using the completeness, we may write the quantum field as follows:

$$\Psi^{\mathrm{op}}=\sum{}'[a_i\psi_{E_i}(\mathbf{x})e^{-tE_it}+b_i\psi_{\bar{E}_i}e^{-i\bar{E}_it}] +\sum{}''[a_i\psi_{Ei}e^{-E_it}+b_i(\psi^a e^{-iE_it}+\psi_{E_i})]+\psi_{\mathrm{normal}}{}^{\mathrm{op}}. \tag{43}$$

Here $\psi_{\mathrm{normal}}{}^{\mathrm{op}}$ is the contribution coming from the continuum and the real-energy bound states without associated eigenvectors and the primes on the summation symbols indicate summing over eigenstates and associated eigenstates, respectively. For the "normal" contribution, the orthogonality proportion in the physical metric, together with the canonical commutation relations, leads to the ordinary (positive-definite metric) commutation relations for the creation and annihilation operators of the particles and antiparticles. In the case of complex eigenvalues, we obtain [using (33)]

$$[a,a^\dagger]=0=[b,b^\dagger]=\cdots, \quad [a,b^\dagger]=i, \tag{44}$$

whereas for the associated situation,

$$[a,b^\dagger]=1. \tag{45}$$

The quantization is the same as explained in Secs. II and III. The Hamiltonian and the charge operator of the canonical theory,

$$H^{\mathrm{op}}=\int\Psi^{\mathrm{op}\dagger}(x)H(x)\Psi^{\mathrm{op}}(x)d^3x,$$

$$Q=-\int\Psi^{\mathrm{op}\dagger}(x)\tau_2\Psi^{\mathrm{op}}(x)d^3x,$$

has (up to c-numbers) the form (13) or (15), where instead of (5) we now have the complex energies E_i or $\bar{E}_i$, respectively.

A special case of such a problem has been discussed by Schiff, Snyder, and Weinberg.[3] For a Klein-Gordon particle in a square-well electric potential, the solution of the c-number problem leads to the following spectrum (Fig. 2). At a value V_{min} of the potential depth (with fixed range), a particle bound state develops. At a larger depth V', an antiparticle bound state appears. Then two bound states coalesce at a value V'', where one encounters an associated eigenvector (dipole-ghost situation). After this value, one has a pair of complex eigenvalues with E and $\bar{E}$. Every bound state develops out of the continuum and runs through the dipole situation before it becomes complex. We differ essentially from the aforementioned authors[3] in the use of associated eigenvectors for reasons of completeness, and in the quantization using an indefinite-metric Fock space, or

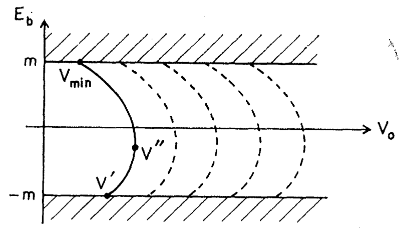

FIG. 2. Bound states as a function of the potential strength in the Schiff-Snyder-Weinberg model.

the "negative oscillator" quantization in a positive-definite Hilbert space without a vacuum. Taking these features into account, there are no inconsistencies or instabilities. The described situation will even remain stable if we add to the stationary interaction a time-dependent interaction with a finite extent in time:

$$H=H_0+V_0+V(x,t). \tag{46}$$

In this case the Yang-Feldman equation can be taken as

$$A(x)=A_{\mathrm{in}}(x)+\int G_R(\mathbf{x},\mathbf{x}';t-t')V(x')d^4x', \tag{47}$$

where G_R is the retarded Green's function of the stationary problem. If the time-dependent potential is extended in time to $t=\pm\infty$, the Yang-Feldman equation would, however, lead to troubles, in view of the fact that G_R has a contribution which exponentially increases in time (coming from the classical complex-energy solution). In the Lee-Wick theory[10] of complex ghosts, one would modify the integration contour in the complex plane, which in our language would correspond to projecting out the complex energies:

$$G_R\to G_R'=G_R-\sum_{\mathrm{Im}E_i\pm 0}\psi_i(\mathbf{x})\bar{\psi}_i(\mathbf{x}')e^{-iE_i(t-t')}. \tag{48}$$

It is evident that this substitution leads to a violation of causality, i.e., the fields $A'(x)$ fulfilling the Yang-Feldman equation with the new Green's function do not commute, i.e.,

$$[A'(x),A'(y)]\neq 0 \quad \text{for} \quad (x-y)^2<0. \tag{49}$$

One can carry this treatment to the case of higher spin, but one should be aware of the aspects of noncausality and their physical implications.[11] The possibility of negative-oscillator quantization does not exist in the case of half-integer spin.

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[10] T. D. Lee and G. C. Wick, Nucl. Phys. **B9**, 209 (1969). Here a prescription is given in terms of the Dyson formula, yielding a unitary S matrix in a positive-definite subspace. In the language of the Yang-Feldman equation and for the special case of external-field models, this is identical with our prescription.

[11] G. Velo and D. Zwanziger, Phys. Rev. **186**, 1337 (1969); **188**, 2218 (1969).

We have been treating in this paper field-theoretical models of the bilinear type:

$$H=H_0+\int V(x)A^\dagger(x)A(x)d^3x. \qquad (50)$$

Realistic interactions are much more complicated trilinear, quadrilinear, or higher couplings. We see no reasons why the phenomenon of indefinite metric and associated eigenstates can not occur if the realistic coupling is sufficiently strong. For the same reasons as one might have overlooked this possibility in the bilinear case, one could be actually ignoring it in the realistic case.

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Finite-Dimensional Spectrum-Generating Algebras

YOSSEF DOTHAN*
Institute for Advanced Study, Princeton, New Jersey 08540
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It is suggested that the generators of a spectrum-generating algebra are all constants of the motion, some of them having an explicit time dependence. Also suggested is a specific form of the Hamiltonian action on the spectrum-generating algebra for systems with a finite number of degrees of freedom. Well-known examples of spectrum-generating algebras are shown to fit into this framework. The stability of the suggested structure against small perturbation is discussed. The question of the generalization of the suggested structure to systems with an infinite number of degrees of freedom is briefly commented upon.

I. INTRODUCTION

SEVERAL years ago the concept of a spectrum-generating algebra (SGA) was introduced[1] as a means of algebraic description of physical systems. This was motivated by the observation that in certain problems[2] series of energy eigenstates with different energies form a basis for a single unitary irreducible representation of a Lie algebra. Thus in a mathematical sense the SGA can be thought of as a generalization of the symmetry algebra (SA). While the SA is represented irreducibly on states which are energy degenerate, a SGA may have as a basis for a single unitary irreducible representation all the energy eigenstates of a system. In fact the SGA was required to have as a subalgebra the SA of the problem.

Of these two algebras the symmetry algebra has an intuitively clear physical definition. Its generators are Hermitian operators which do not have an explicit time dependence and satisfy the following conditions.

(a) They commute with the Hamiltonian of the problem. Since they do not have an explicit time dependence they are constants of the motion.

(b) They form a Lie algebra under commutation. Namely, the commutator of two generators is a linear combination of the generators of the algebra with coefficients which are numbers.

(c) The symmetry algebra is maximal in the sense that for any energy eigenvalue the space of all degenerate states is irreducible under the algebra. This means that we do not allow "accidental" degeneracies. (Since we discuss the symmetry algebra and not the symmetry group, we have to exclude from the discussion degeneracies explainable only by discrete symmetries. However, it is easy to generalize the conditions to symmetry groups instead of symmetry algebras).

(d) The symmetry algebra is minimal in the sense that it does not have a proper subalgebra with the same properties.

On the other hand, the definition of the SGA is more mathematical. One searches for a Lie algebra of Hermitian generators which has the symmetry algebra as a subalgebra such that all the energy eigenfunctions of the physical problem which satisfy the same boundary conditions form a basis for a single unitary irreducible representation of the algebra. This may be considered a generalization of conditions (b) and (c) above. Condition (d) has an obvious generalization, but condition (a) is not generalized. Stated differently, one poses a problem of embedding all the spaces of states which are irreducible under the symmetry algebra in a space

* On leave from the Department of Physics and Astronomy, Tel-Aviv University, Tel-Aviv, Israel.

[1] Y. Dothan, M. Gell-Mann, and Y. Ne'eman, Phys. Rev. Letters **17**, 145 (1965); Y. Dothan and Y. Ne'eman, in *Proceedings of the Second Topical Conference on Resonant Particles*, edited by B. A. Munir (Ohio U. P., Athens, Ohio, 1965), p. 17. The same concept is also known as a noninvariance group or a dynamical group. See, e.g., N. Mukunda, L. O'Raifeartaigh, and E. C. G. Sudarshan, Phys. Rev. Letters **15**, 1041 (1965); A. O. Barut and A. Böhm, Phys. Rev. **139**, B1107 (1965).

[2] It is interesting that in most of the classical analogs of these problems the motion is completely degenerate in the classical sense. Namely, the motion is simply periodic instead of being multiply periodic. See, e.g., H. Goldstein, *Classical Mechanics* (Addison Wesley, New York, 1959), p. 297.